Chapter 5
Integration

5.1 Antiderivatives and Indefinite Integration

Objectives: By the end of this section student should be able to:

- Find Antiderivatives
- Use indefinite integral notation for Antiderivatives
- Use Basic Integration Rules to fine Antiderivatives

Definition: (Antiderivative)

A function **F** is an antiderivative of a function f on an interval **I**, if $F'(x) = f(x)$ for all *x* in **I**.

Example1: Find **an antiderivative** for each of the following.

a) $f(x) = 0$
b) $f(x) = 1$
c) $f(x) = x$
d) $f(x) = x^2$
e) $f(x) = x^3$
f) $f(x) = x^n$
g) $f(x) = \sin x$
h) $f(x) = \cos x$
i) $f(x) = e^x$
j) $f(x) = 1/x$

Definition:

The family of antiderivatives of a function f is denoted by the notation $\int f(x)dx$.

Indefinite Integral

$\int f(x)dx$ is called the indefinite integral of f **.** If F is an antiderivative of f, then $\int f(x)dx = F(x) + C$, where **C** is an arbitrary constant.

Example 2: Find the **definite integral**

a) $\int 0\, dx$
b) $\int 1\, dx$
c) $\int x\, dx$
d) $\int x^2\, dx$
e) $\int \sin x\, dx$
f) $\int \cos x\, dx$

Integration Rules

1) Power Rule: For any **real number** $\boldsymbol{n} \neq \mathbf{1}$. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

Example 3: Find the anti-derivative of each

a) $\int x^3 dx$

b) $\int x^4 dx$

c) $\int \frac{1}{x^2} dx$

d) $\int \sqrt{x} dx$

e) $\int dx$

2) Constant Rule: For any constant $\boldsymbol{k}$ $\int k \ f(x)dx = k\int f(x)dx$

Example 4: Find the anti-derivative of each.

a) $\int 18x^8 dx$

b) $\int 5dx$

c) $\int 9x^{-5} dx$

3) Sum or Difference Rule: $\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)\,dx$

Example 5: Find the anti-derivative of each.

a) $\int (5x^2 - 6x + 3)dx$

b) $\int (4x^{11} - 7x^3 + 4)dx$

c) $\int x(x^2 - 3)dx$

d) $\int 12x^3\sqrt{x}dx$

e) $\int \frac{\sqrt{x}+3}{x^2}dx$

4) Rules for Exponential and Logarithmic Functions

a) $\int e^x dx = e^x + C$

b) $\int e^{kx} dx = \frac{1}{k}e^x + C, \quad k \neq 0$

c) $\int x^{-1} dx = ln\,|\,x\,| + C$

Example 6: Find the anti-derivative for each.

a) $\int 3e^{x}dx$

b) $\int e^{2x}dx$

c) $\int \frac{x^4+1}{x}dx$

d) $\int x^3\left(x+x^{-4}\right)dx$

e) $\int (e^{2x}+4x)dx$

Example 7: Find the ***Profit*** function ***P(x),*** if $\boldsymbol{P'(x) = 4x - 5}$ and $\boldsymbol{P(1) = 4.}$

Recall that if the function s(t) gives the position of a particle at time t, then its velocity v(t) and its acceleration a(t) are given by: $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

Example 8: For a particular object in motion: $\boldsymbol{a(t) = t^2 + 1}$ and $v(0) = 6$.
Find $\boldsymbol{v(t).}$

Integration by Substitution

Example: Evaluate the following indefinite Integrals

a) $f(x) = \int (3x-5)^2 dx$

b) $f(x) = \int (3x-5)^5 dx$

Theorem (Integration by Substitution)

If an integral is given in the form $\int f(g(x)) \cdot g'(x) dx$, then we can use the substitution $u = g(x)$ and the given integral can be reduced to the form:

$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$

Note: Making the substitution $u = g(x)$ usually makes it easy to find the antiderivative of the original integral

Example 1: Evaluate the following indefinite Integrals using the substitution method

a) $\int (3x-5)^5 dx$

b) $\int 6(3x+5)^3 dx$

c) $\int (x^2-1)e^{x^3-3x} dx$

d) $\int z^2\sqrt{z^3+12}\ dx$

e) $\int \frac{2y}{y^2+1} d$

f) $\int \frac{4x+3}{2x^2+3x+1} dx$

g) $\int \frac{2}{5y+9} dx$

Example 2: Suppose the marginal revenue (in dollars) from the sale of x jet planes is given by $R'(x) = 2x(x^2+50)^2$. Find the **total revenue function** $R(x)$ if the revenue from **3 planes** is $206,379.

General Rule for Substitution Method

In general, for the types of problems we are concerned with, there are three cases. We choose ***u*** ***or some other variable*** **to be one of the following:**

1. The quantity under a root or raised to a power;
2. The exponent of e;
3. The expression in the denominator.

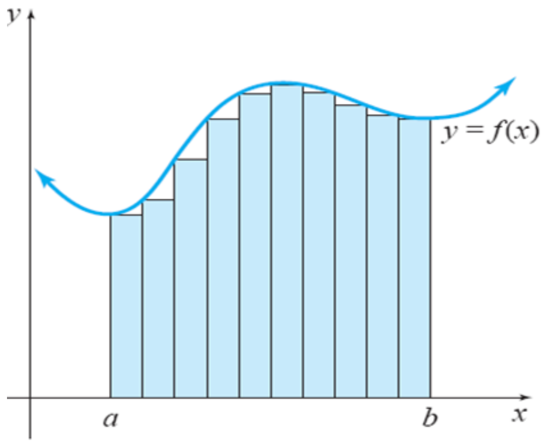
Area and Definite Integral

In this section we are going to find and approximate the area under a curve and above the x-axis.

Consider the region **R** bounded by the graph of the function $y = f(x)$, the **x – axis** on the interval **[a, b].**

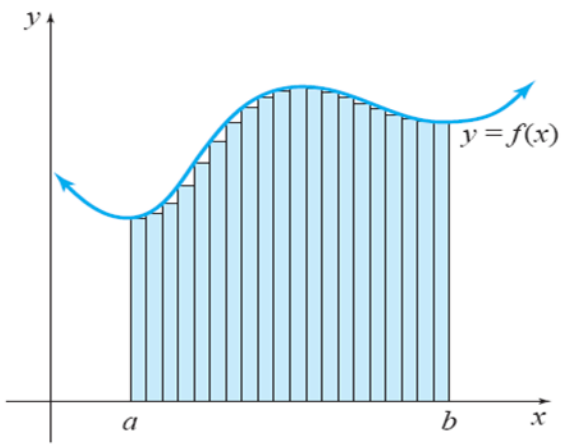
We want to **Find (estimate)** the area of the region **R** on $[a,\ b]$**. See graphs below**

a) Area approximated by 10 rectangles



Ten rectangles of equal width

b) Area approximated by 20 rectangles



Twenty rectangles of equal width

Note: The more the rectangles the better the approximation for the area of the region **R**

In case there are **n** rectangles; the total area under the curve is approximated by the sum of the areas of the n – rectangles. With sigma notation the approximation to the total area becomes:

Area of the n – rectangles $= \sum_{i=1}^{n} f(t_i)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$ is the change in x and $f(t_i)$ is the height of the rectangle at the ***x* – value** t_i

The **exact area** is defined to be the limit of this sum (if the limit exists) as the number of rectangles n increases without bound.

$$\textbf{Exact Area} = \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(t_i)\Delta x_i\right)$$

Note: The Sum $\sum_{i=1}^{n} f(t_i)\Delta x_i$ is called the **Riemann Sum** for the function f on the interval **[a, b]**

Definition (The Definite Integral)

If f is defined on an interval $[a, b]$, the **definite integral** of f from a to b is given by:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \left(\sum_{i=1}^{n} f(t_i)\Delta x\right)$$

provided this limit exists, where $\Delta x = (b - a)/n$ and t_i is any value of x in the i^{th} interval $[x_{i-1}, x_i]$.

Example 3: Approximate the area under the curve and above the x-axis using n rectangles. Let the height of each rectangle be given by the value of the function at the left side of the rectangle.
$f(x) = 2x + 3$ from x = 0 to x = 2; n = 4

Example 4: Approximate the area under the curve and above the x-axis using n rectangles. Let the height of each rectangle be given by the value of the function at the right side of the rectangle.
$f(x) = 2x^2 + x + 3$ from x = 0 to x = 6; n = 6

Example 5: What is the exact value of the area under the curve in Ex 3 and Ex 4?

Example 5: Calculate the **Riemann Sum** for the given integral consider midpoints in the subintervals: with n = 6 for the integral $\int_3^6 (4x - 7)dx$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus makes evaluation of the definite integral much easier. The Fundamental Theorem of Calculus establishes a connection between the two branches of calculus: Differential Calculus and Integral Calculus. The Fundamental Theorem of Calculus has two parts.

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,\ b]$, then the function g defined by $g(x) = \int_a^x f(t)dt$ $a \le t \le b$ is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$, and $g'(x) = f(x)$

Proof: Refer to the Book

Example 1: Find the derivative of each of the following function.

a) $g(x) = \int_0^x \sqrt{1+t^2}dt$

b) $f(x) = \int_1^{x^4} sec\ t\ dt$

c) $y = \int_1^{\cos x}(t + \sin t)\ dt$

The Fundamental Theorem of Calculus, Part 2

If f is continuous on $[a, b]$ and F is any **antiderivative** of f, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof: Refer to Text Book

Example 2: Evaluate the following integrals

a) $\int_{-1}^2 x^3 dx$

b) $\int_\pi^{2\pi} \sin(2x)\ dx$

c) $\int_{-4}^2 \frac{2}{x^6} dx$

The Fundamental Theorem of Calculus: Suppose f is continuous on $[a,\ b]$

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
2. If F is any **antiderivative** of f, then $\int_a^b f(x)dx = F(b) - F(a)$

Note: If f is continuous and non-negative on $[a, b]$, then $\int_a^b f(x)dx$ is the area of the region **R** bounded by the graph of the function $y = f(x)$, the **x – axis** on the interval **[a, b]**

Example 3: Evaluate each definite integral:

a) $\int_0^2 (2x+3)dx$

b) $\int_0^4 x^2 dx$

Properties of Definite Integrals:

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$

3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Example 4: Evaluate the following integrals

a) $\int_{-1}^2 (3t-1)dt$

b) $\int_0^6 (2x^2+x+3)dx$

c) $\int_1^e \frac{1}{x} dx$

d) $\int_0^3 e^{2x} dx$

e) $\int_1^4 \frac{t^3+1}{\sqrt{t}} dt$

f) $\int_3^5 \frac{x^2-9}{x+3} dx$

g) $\int_1^5 (5n^{-2} + n^{-3})dn$

Example 4: Find the definite integral (integrals that require substitution)

a) $\int_0^2 x(x^2+3)^4 dx$

b) $\int_0^1 3x^2 e^{x^3} dx$

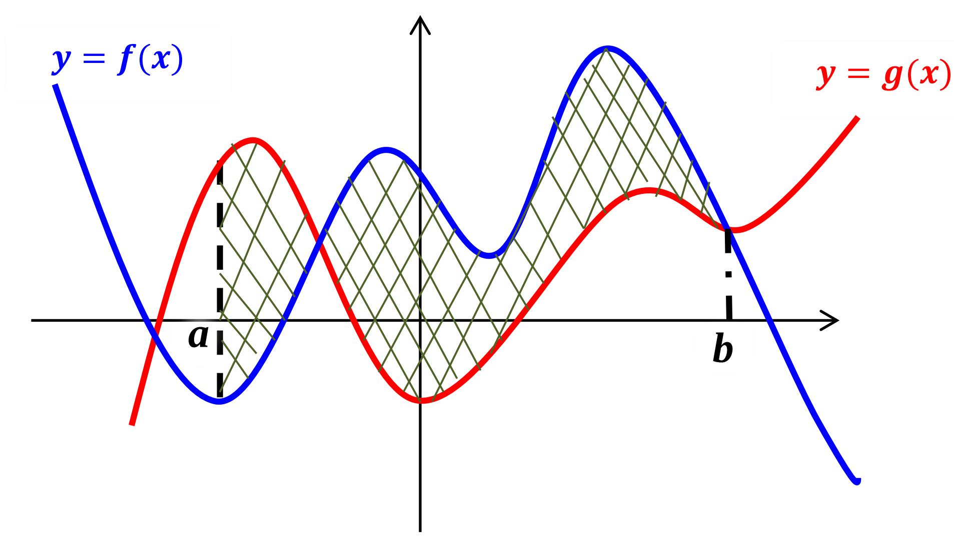
c) $\int_0^1 5x \cdot \sqrt[3]{1+x^2}\, dx$

Area between Two Curves

In the last section, we found the area under the curve and above the x-axis. In this section we will expand it to finding the area when we have 2 curves.

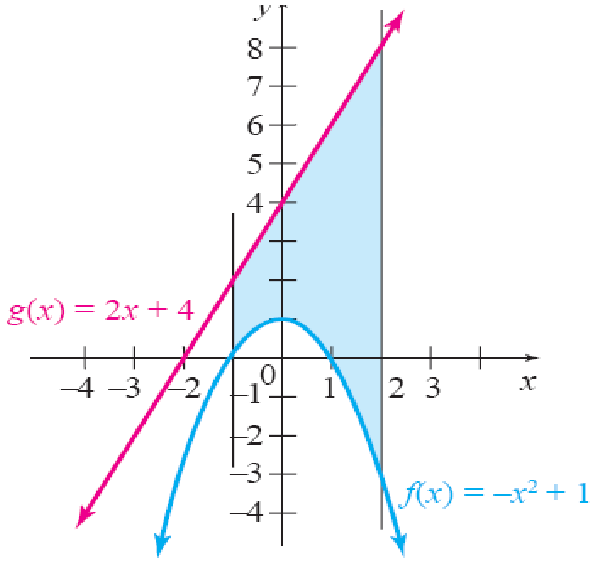
Let be $y = f(x)$ *and* $y = g(x)$ functions whose graphs are shown in the figure below. The area A of the shaded region bounded by the two functions on the interval [a, b] is given by:

$$A = \int_a^b |f(x) - g(x)| dx = \int_a^b [\text{Top function} - \text{Bottom function}] dx$$



Example 1: Find the area between $y = x^2 + 4$ and $y = 0$ on the interval **[-2, 1]**

Example 2: Find the area between the $f(x) = -x^2 + 1$ and $g(x) = 2x + 4$ between $x = -1$ and $x = 2$; **see figure** below



Example 3: Find the area between each of the following curves.

a) $f(x) = 5 - x^2,\ g(x) = x^2 - 3$

b) $f(x) = x^2 - 5x + 4,\ g(x) = -(x-1)^2$

c) $y = x^3,\ y = x$

a) $f(x) = x^2 - 30,\ g(x) = 10 - 3x$